PRECORSO DI MATEMATICA 2021/2022 – CLASSE N

LEZ. 2
30/09/2021

Funzioni A e B insiemi non vuoti

Una *funzione* $f : A \to B$
è una relazione che associa
ad ogni elemento di A uno e un solo
elemento di B.

A dominio , B codominio

L'immagine di A tramite f è l'insieme

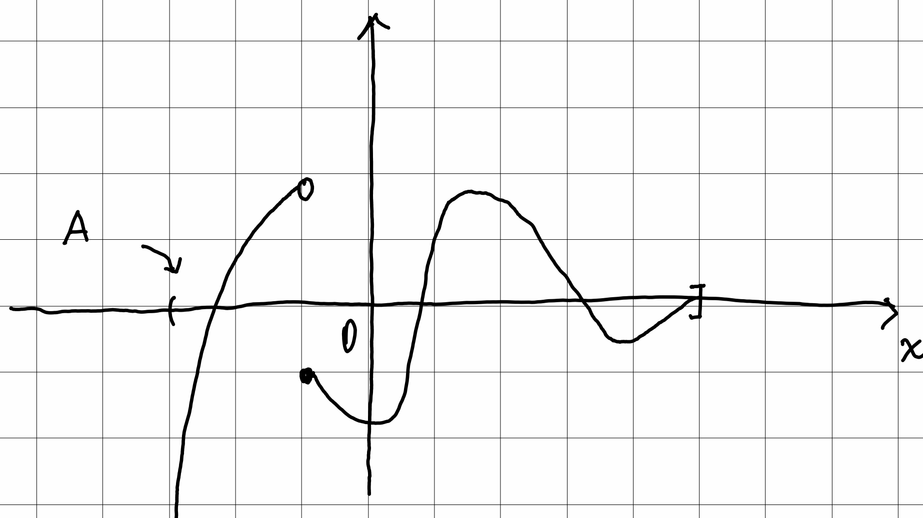
$$f(A) := \{ y \in B : \exists\, x \in A \text{ t.c. } f(x) = y \}$$

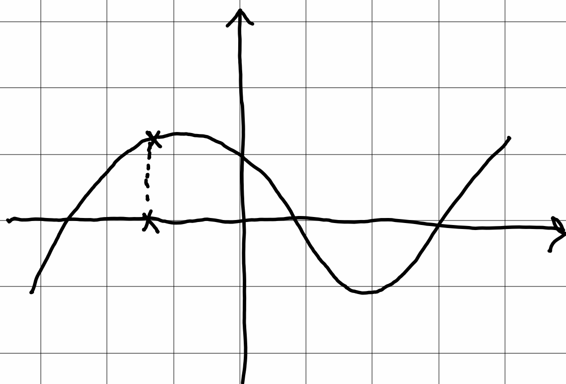
Si usa anche la notazione $x \in A$, $x \mapsto y$

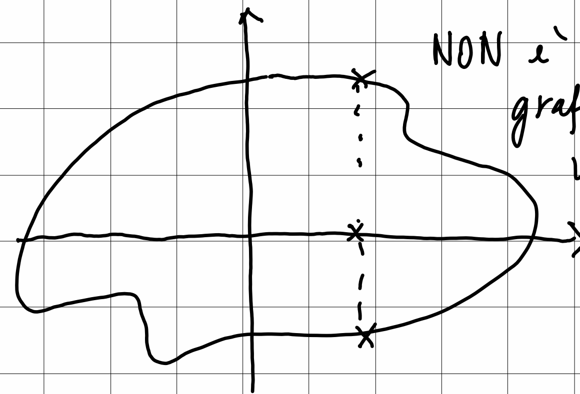
Funzioni $\underbrace{\text{reali}}_{B = \mathbb{R}}$ $\underbrace{\text{di variabile reale}}$ vuol dire che il dominio $A \subset \mathbb{R}$

$$f : A \longrightarrow \mathbb{R}$$

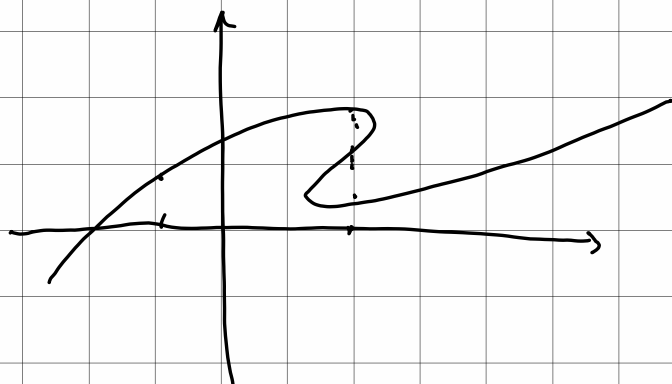
Grafico di funzione reali di variabile reale







NON è il grafico di una funzione



Esempi : $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$x \longmapsto x \qquad f(x) = x$$

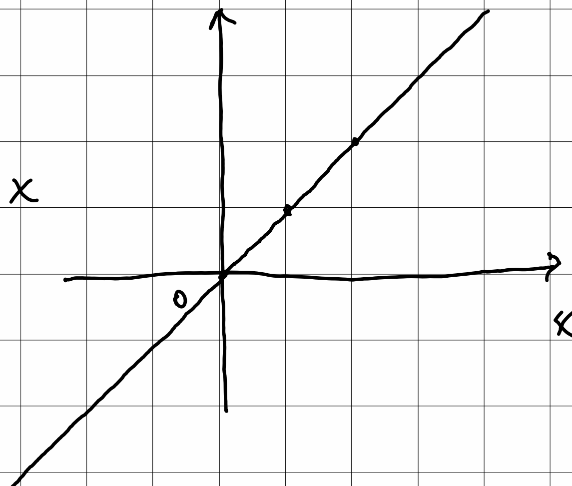
$$f : [0, 1] \longrightarrow \mathbb{R}$$
$$f(x) = x$$

Determinare il dominio della funzione

$$f(x) = x + 2$$

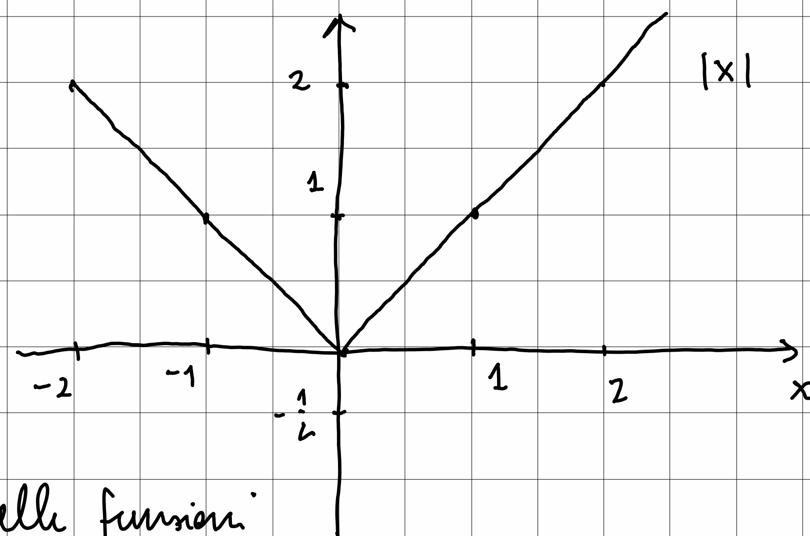
$$f(x) = x$$

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = x$$



Esempio: $f(x) = |x| = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x < 0 \end{cases}$

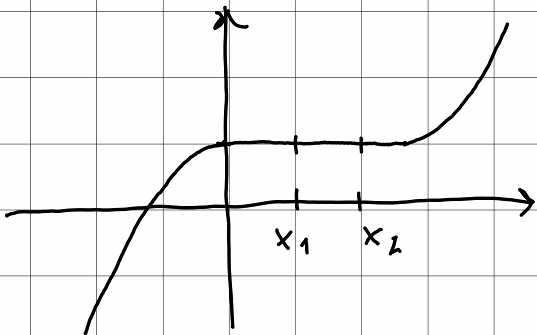
$f : \mathbb{R} \to \mathbb{R}$



Monotonia delle funzioni

Def: $f : A \to \mathbb{R}$ è crescente (risp. strettamente crescente) se $\forall\, x_1, x_2 \in A$, $x_1 \leq x_2$ (risp. $x_1 < x_2$) risulta $f(x_1) \leq f(x_2)$ (risp. $f(x_1) < f(x_2)$)

$\rightsquigarrow$ decrescente: $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$



f è strettamente crescente

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

I CASO: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \Rightarrow f(x_1) \neq f(x_2)$

II CASO: $x_2 < x_1 \Rightarrow f(x_2) < f(x_1) \Rightarrow f(x_1) \neq f(x_2)$

Def: f è <u>iniettiva</u> se $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$f: A \to B$ A, B insiemi non vuoti.

Def: f è **suriettiva** se $f(A) = \mathbb{R}$

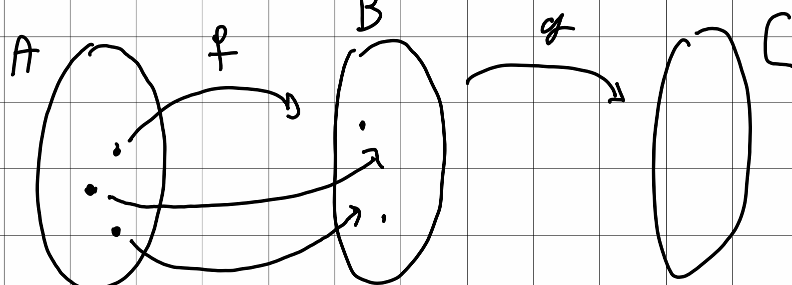
($f: A \to B$ $\quad f(A) = B$)

Def: f è **biettiva** se è iniettiva e suriettiva

Quindi $\forall y \in \mathbb{R} \ \exists! \ x \in A$ t.c. $f(x) = y$

Def: A, B, C insiemi non vuoti

$f: A \to B \qquad g: B \to C$



Composta: $g \circ f : A \to C \qquad g \circ f(x) := g(f(x))$

$\forall x \in A$

Def: $f: A \to B$ è **invertibile** se $\exists\, g: B \to A$ t.c. $\forall x \in A$: $g(f(x)) = x$

$\forall y \in B$: $f(g(y)) = y$

$$g \circ f = id \quad e \quad f \circ g = id$$

Se $f: A \to B$ è biettiva

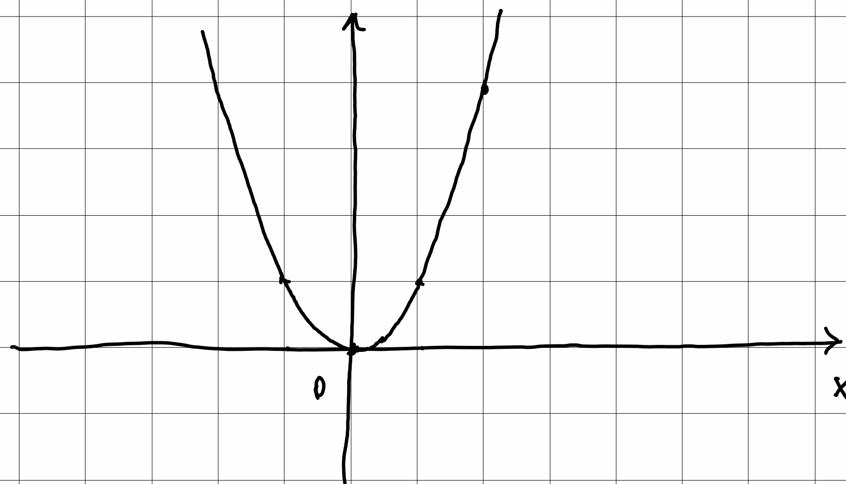
$$\forall y \in B \ \exists!\, x \in A \ \text{t.c.}\ f(x) = y$$

$g: B \to A \qquad \forall y \in B \quad g(y) := x$ dove x è l'unico elemento di A t.c. $f(x) = y$

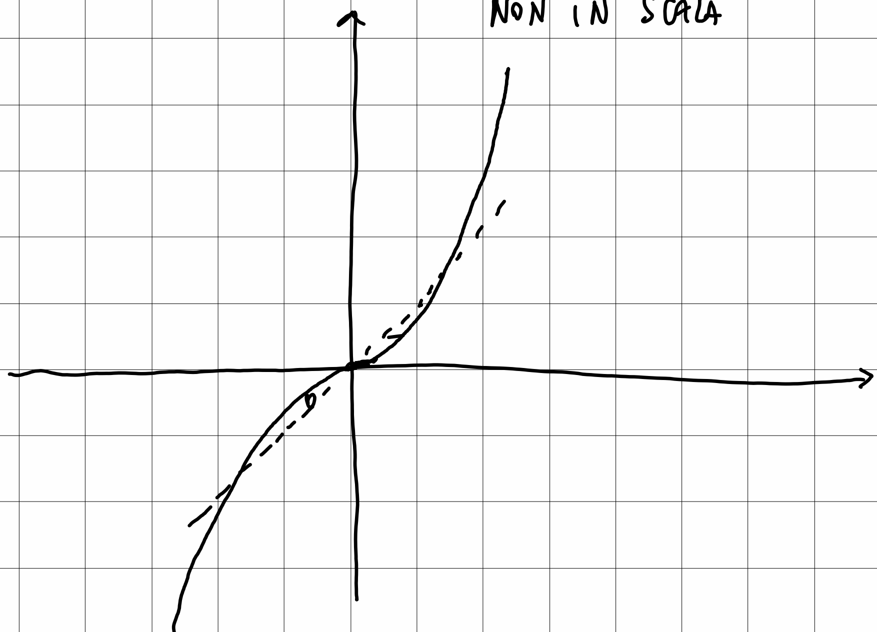
$$x = g(y) = g(f(x))$$

__Esempi__: $f(x) = x^2$

$f: \mathbb{R} \to \mathbb{R}$



- $f(x) = x^3$

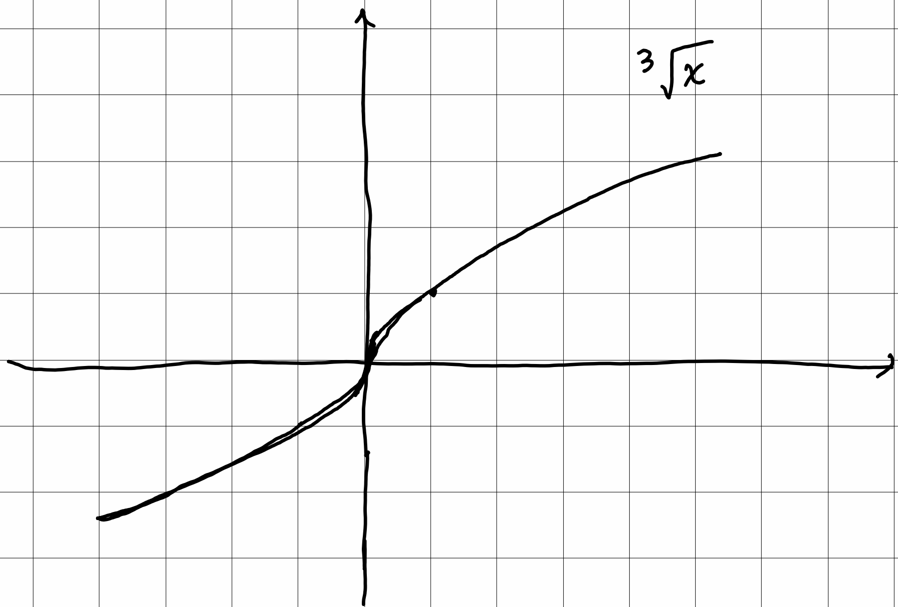


$g: \mathbb{R} \to \mathbb{R}$ inversa di $f(x) = x^3$

$g(y) = x$ quando $f(x) = y$ cioè $x^3 = y$

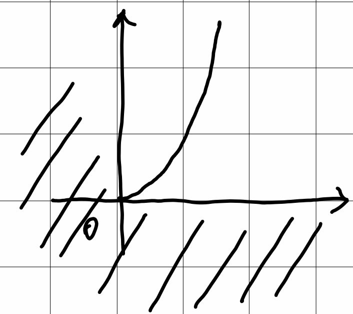
$x = \sqrt[3]{y}$

$g(y) = \sqrt[3]{y}$


$\sqrt[3]{x}$

- $f(x) = x^2 \qquad f: \mathbb{R} \to \mathbb{R}$

$f^*: [0, +\infty) \longrightarrow [0, +\infty)$

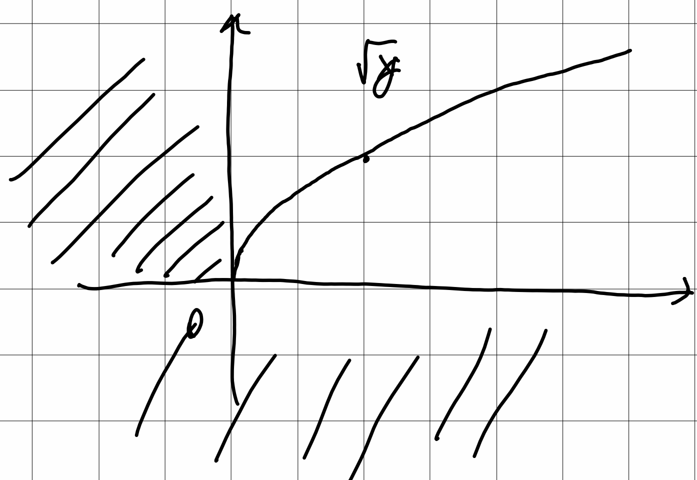


$g: [0, +\infty) \longrightarrow [0, +\infty)$

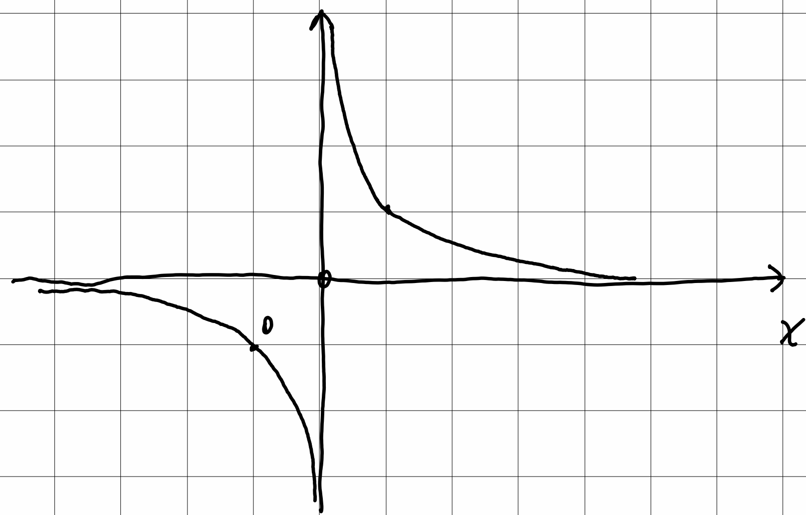
$g(y) = x$ quando $f(x) = y$ con $x \in [0, +\infty)$

$x^2 = y$

$x = \sqrt{y}$



• $f(x) = \frac{1}{x} = x^{-1}$ $\quad f: \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$



• x^α $\alpha \in \mathbb{R}$ $f(x) = x^\alpha$

I CASO : $\alpha = 1$ visto $f(x) = x$

II CASO : $\alpha > 1$ $x \geq 0$

$\alpha - 1 > 0$ $0 \leq x < 1$

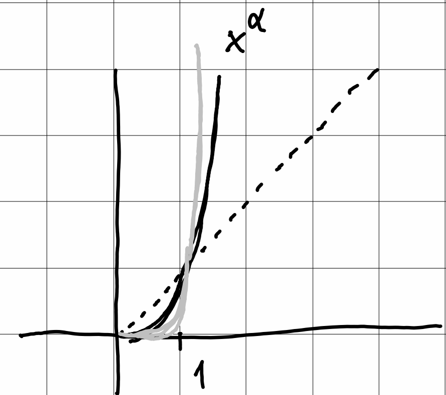
$x^{\alpha-1} < 1$

$x^\alpha < x$

$1 < x$

$1 < x^{\alpha-1}$

$x < x^\alpha$



$\alpha < \beta$ $\beta - \alpha > 0$

$0 \leq x < 1$

$x^{\beta-\alpha} < 1$

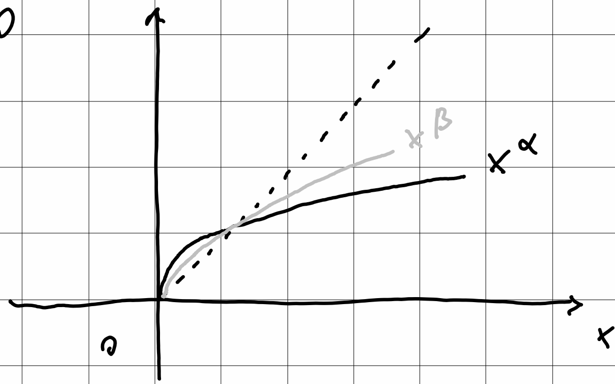
$x^\beta < x^\alpha$

$1 < x$ $1 < x^{\beta-\alpha}$

$x^\alpha < x$

III caso: $0 < \alpha < 1$

$1 - \alpha > 0$



se $0 \leq x < 1$

$x^{1-\alpha} < 1$

$x < x^\alpha$

se $x < 1$

$x^\alpha < x$

$\alpha < \beta$ $\quad \beta - \alpha > 0$

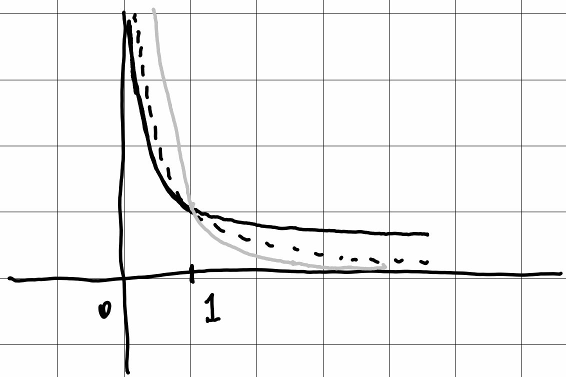
$0 \leq x < 1$

$x^{\beta - \alpha} < 1$

$x^\beta < x^\alpha$

$x^{-\alpha} = \frac{1}{x^\alpha}$ definito per $x > 0$

IV CASO : $\alpha = 1$: $\frac{1}{x}$



V CASO: $0 < \alpha < 1$ $1 - \alpha > 0$

se $0 < x < 1$

$x^{1-\alpha} < 1$

$x < x^\alpha$ $\frac{1}{x} > \frac{1}{x^\alpha}$

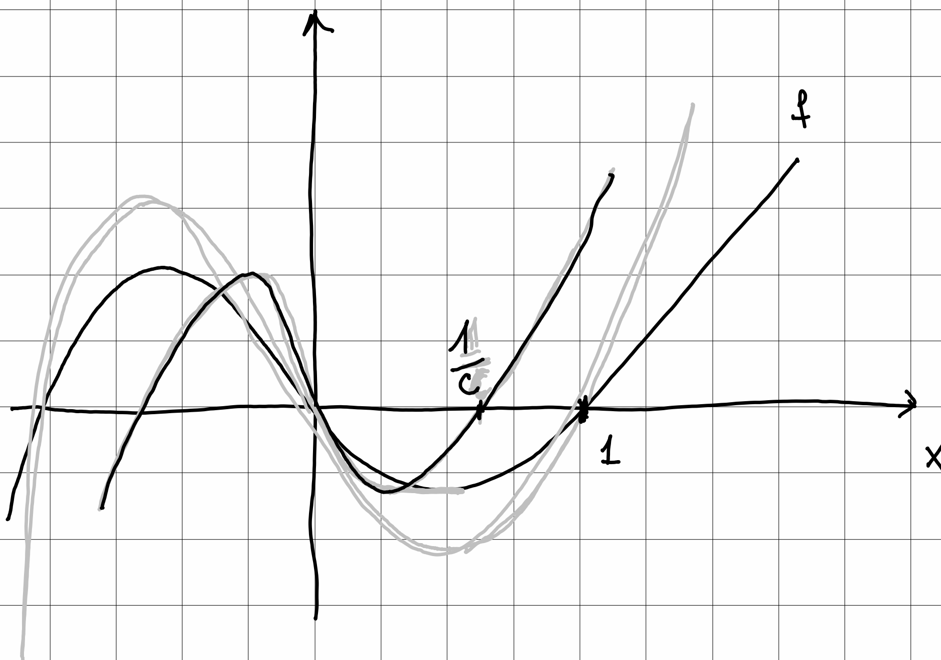
se $1 < x$

$\frac{1}{x} < \frac{1}{x^\alpha}$

VI CASO: $1 < \alpha$ $\alpha - 1 > 0$

$0 < x < 1$ $x^{\alpha - 1} < 1$ $x^\alpha < x$ $\frac{1}{x} < \frac{1}{x^\alpha}$

Operazioni sui grafici



$f(x) + c$

$f(x - c)$

$|f(x)|$

$f(|x|)$

$c\,f(x)$

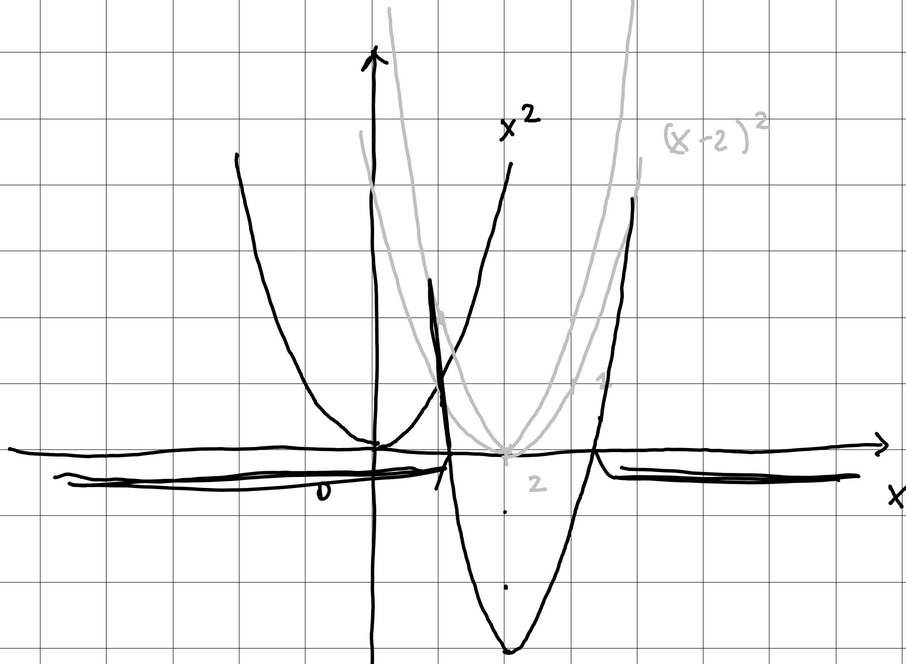
$f(cx)$

$cx = 1$

$x = \frac{1}{c}$

Esempi:

- $2(x-2)^2-3 \geq 0$



POLINOMI

Def: Una funzione polinomimiale (<u>polinomio di grado n</u>) è $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

dove $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ per $i = 0, \dots, n$, $a_n \neq 0$

$\deg(0) := -1$

$P(x)$ $\quad \deg(P) = n$

P e Q sono polinomi $\quad P + Q \quad \deg(P+Q) \leq$
$\deg P = n \quad \deg Q = m \quad \leq \max\{\deg P, \deg Q\}$

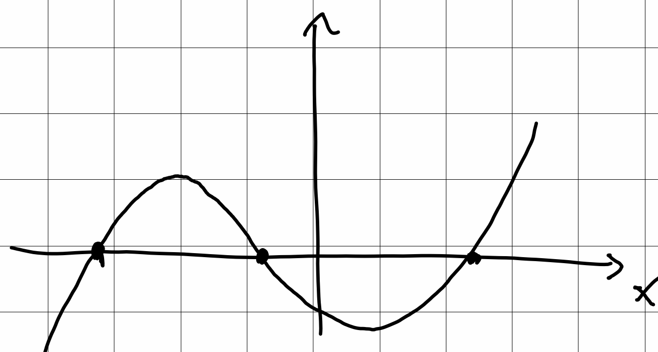
$x^2 + 1$ e $-x^2 + x \rightsquigarrow x + 1$

$$\deg(P \cdot Q) = \deg P + \deg Q$$

$$P(x) = a_n x^n + \dots$$
$$Q(x) = a_m x^m + \dots$$

$$P(x)Q(x) = a_n \cdot a_m \, x^{n+n} + \dots$$

<u>Def</u>: $z \in \mathbb{R}$ è una <u>radice</u> di $P(x)$ (oppure uno <u>zero</u> di $P(x)$) se $P(a) = 0$.



Esempio: 2 è radice di x^2-3x+2

$$4-6+2=0 \quad \checkmark$$

POLINOMI DI II GRADO

$$(x-z)^2-k$$

$$ax^2+bx+c= \qquad a\neq 0\ ,\ a>0 \qquad (x+z)^2=$$

$$=x^2+2xz+z^2$$

$$=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)$$

$$=a\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}\right)-\frac{b^2}{4a}+c=$$

$$=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}$$

Questo è $=0$ se e solo se $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

$b^2-4ac=\Delta$ discriminante

• Se $\Delta = b^2 - 4ac \geq 0$

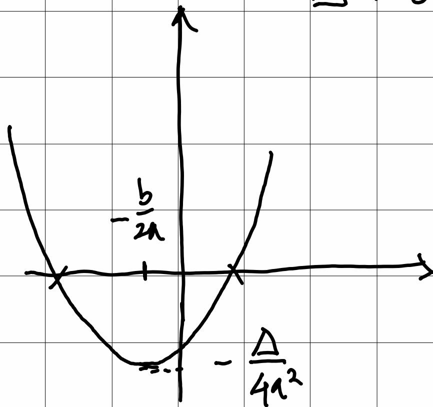
$$x + \frac{b}{2a} = \frac{\sqrt{\Delta}}{2a} \quad \text{oppure} \quad x + \frac{b}{2a} = -\frac{\sqrt{\Delta}}{2a}$$

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{oppure} \quad x = \frac{-b - \sqrt{\Delta}}{2a}$$

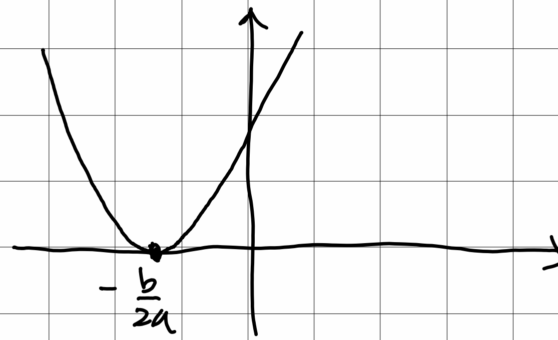
Osserviamo: le due radici sono distinte quando $\Delta > 0$, altrimenti, se $\Delta = 0$, c'è una radice doppia $-\frac{b}{2a}$

• Se $\Delta < 0$ non ci sono radici

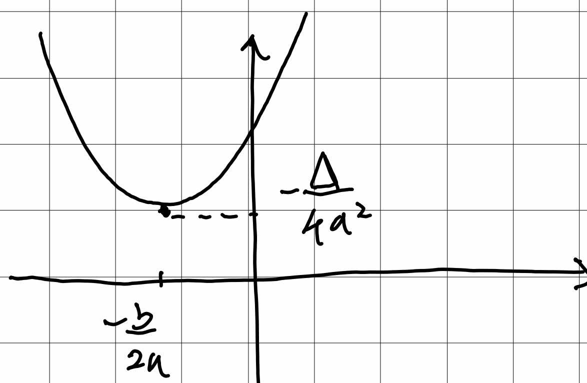
$\Delta > 0$



$\Delta = 0$



$\Delta < 0$



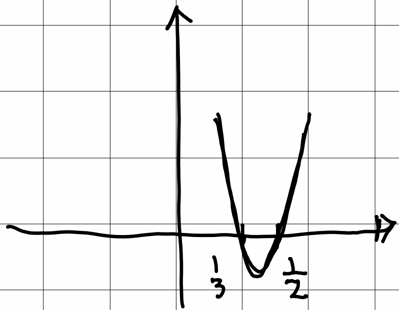
Esercizi:

$$6x^2 - 5x + 1 \le 0$$
$$6x^2 - 3x - 2x + 1 \le 0$$
$$3x(2x-1) - (2x-1) \le 0$$
$$(2x-1)(3x-1) \le 0$$

$$(x+a)(x+b) = x^2 + (a+b)x + ab$$
$$(x+a)(x-a) = x^2 - a^2$$
$$(x+a)^2 = x^2 + 2ax + a^2$$

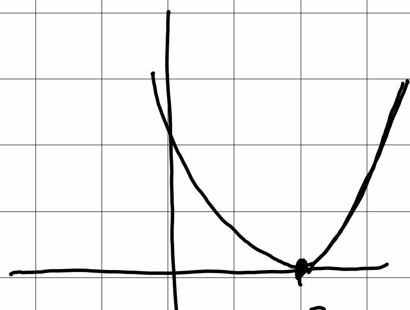
$x = \frac{1}{2}$ $\quad x = \frac{1}{3}$ sono radici $\quad \Delta > 0$

$$\frac{1}{3} \le x \le \frac{1}{2}$$



- $x^2 - 4x + 4 > 0$

$(x-2)^2 > 0$ $\qquad x \neq 2 \qquad x \in \mathbb{R} \setminus \{2\}$

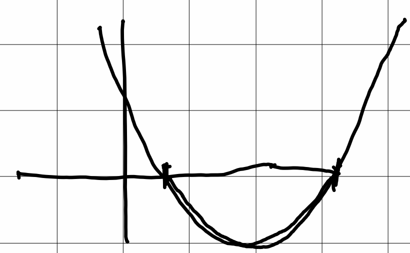


- $x^2 - 5x + 1 < 0$

$\Delta = b^2 - 4ac = 25 - 4 = 21 > 0 \qquad \sqrt{21}$

radici $x = \dfrac{5+\sqrt{21}}{2}$ e $x = \dfrac{5-\sqrt{21}}{2}$

$\dfrac{5-\sqrt{21}}{2} < x < \dfrac{5+\sqrt{21}}{2}$



- $2x^2 + \pi x + 2 < 0$

$$\Delta = \pi^2 - 16 < 0$$

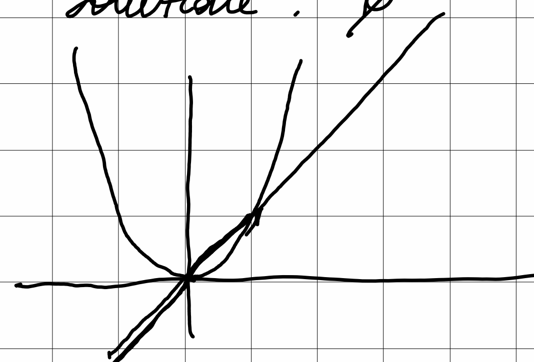
$$\sqrt{\pi^2} = \pi \qquad \sqrt{16} = 4$$

$$\pi < 4$$

$$\pi^2 < 16$$

Non è verificata per alcun x

Soluzione : $\emptyset$.



- $x^2 - x \geq 0$

$$x^2 \geq x$$

$x \leq 0$ oppure $x \geq 1$

POLINOMI DI GRADO SUPERIORE

- $16x^4 - 8x^2 + 1$ che radici ha?

$y = x^2$ $\qquad$ $16y^2 - 8y + 1 =$

$= (4y-1)^2 = 0$ per $4y - 1 = 0$

$y = \frac{1}{4}$

$x^2 = \frac{1}{4}$ $\qquad$ $x = \frac{1}{2}$ oppure $x = -\frac{1}{2}$

- $x^5 + x^3 + 8x^2 + 8 = x^3(x^2+1) + 8(x^2+1) =$
$= (x^2+1)(x^3+8) = (x^2+1)(x+2)(x^2-2x+4)$

$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$

- $x^9 - 4x^7 - x^4 + 4x^2 \geq 0$

$$x^7(x^2-4) - x^2(x^2-4) \geq 0$$

$$(x^7 - x^2)(x^2-4) \geq 0$$

$$x^2(x^5-1)(x-2)(x+2) \geq 0$$

$$x^5 - 1 \geq 0 \qquad x^5 \geq 1$$

$$x \geq 1$$

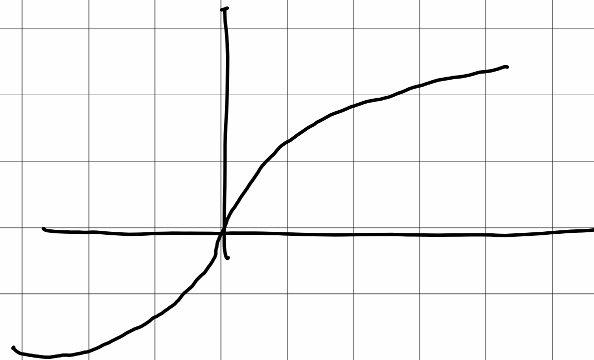
	-2		1		2	
(x^5-1)	$-$	$-$		$+$		$+$
$(x-2)$	$-$	$-$		$-$		$+$
$(x+2)$	$-$	$+$		$+$		$+$
	$-$	$+$		$-$		$+$

$$[-2, 1] \cup [2, +\infty)$$

- $x^3 - 8 \geq 0$

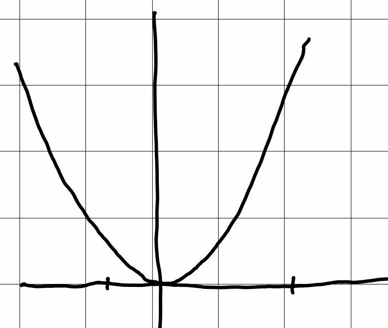
$x^3 \geq 8$

$x \geq 8$



- $(2x-3)^4 < (1-x)^4$

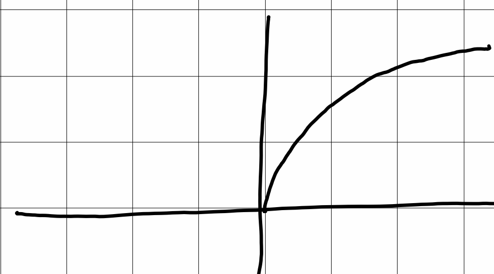
$((2x-3)^2)^2 < ((1-x)^2)^2$



$(2x-3)^2 < (1-x)^2$

NO: $2x-3 < 1-x$

$\sqrt{y^2} = |y|$



$$|2x-3| < |1-x|$$

		1		$\frac{3}{2}$	
$(2x-3)$	$-$		$-$		$+$
$(1-x)$	$+$		$-$		$-$

$1-x \geq 0$

Soluzione:

$\left(\frac{4}{3}, 2\right)$

I case: $x < 1$

$-2x+3 < 1-x \qquad 2 < x \qquad$ Nessuna

II case: $1 \leq x < \frac{3}{2}$

$-2x+3 < x-1$

$4 < 3x \qquad \frac{4}{3} < x < \frac{3}{2}$

$\frac{4}{3} - \frac{3}{2} = \frac{8-9}{6} < 0 \qquad \frac{4}{3} < \frac{3}{2}$

III case: $\frac{3}{2} \leq x$

$2x-3 < x-1 \qquad \frac{3}{2} \leq x < 2$

- $x^4 + 2x^3 + 6x^2 + 5x - 14 \geq 0$

$*$ con coefficienti $a_i \in \mathbb{Z}$

<u>Teorema</u> (delle radici razionali): Sia

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

un polinomio di grado n.* Sia $\frac{p}{q} \in \mathbb{Q}$ con $\frac{p}{q}$ ridotta ai minimi termini (p e q non hanno divisori comuni). Supponiamo che $\frac{p}{q}$ sia radice di $P(x)$. Allora p divide a_0 e q divide a_n.

<u>Dim</u>: $P\left(\frac{p}{q}\right) = 0$

$$a_n \frac{p^n}{q^n} + a_{n-1} \frac{p^{n-1}}{q^{n-1}} + \dots + a_1 \frac{p}{q} + a_0 = 0$$

$$\underbrace{a_n p^n + a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1}} + a_0 q^n = 0$$

$s \in \mathbb{Z}$ $\qquad p \cdot s = -a_0 q^n \Rightarrow p$ divide a_0 $\square$

$$x^4 + 2x^3 + 6x^2 + 5x - 14$$

$1: \quad 1 + 2 + 6 + 5 - 14 = 0 \quad \checkmark$

$-1: \quad 1 - 2 + 6 - 5 - 14 \neq 0$

$2: \quad 16 + 16 + 24 + 10 - 14 \neq 0$

$-2: \quad \cancel{16} - \cancel{16} + 24 - 10 - 14 = 0 \quad \checkmark$

$7: \quad \dots$

$-7: \quad \dots$

$$P(x) = (x-1)(x+2)\underbrace{Q(x)}$$

Divisione: $P(x)$ diviso $Q(x)$